Supplemental Appendix to “Banks, Liquidity Crisis and Economic Growth”

In this document we present the supplementary proofs of "Banks, Liquity Crises and Economic Growth". The proofs included are the solutions to the distinct financial arrangements: (1) Financial Autarky, (2) The unconstrained optimal risk sharing, (3) covered banking and, (4) exposed banking.

1 Financial Autarky

1.1 The Autarkic Solution

Under financial autarky agents realize their invetsment decisions by themselves and cannot share their liquidity risk. Consumption in the two states of the liquidity shock is determined by their investment choice: $c_E = w - k + \gamma h(k)$ and $c_L = w - k + h(k)$. Therefore the value function is : $V(k,w) = \pi u(w - k + \gamma h(k)) + (1-\pi) u(w - k + h(k))$.

The optimal program is to find k_{opt} that maximizes $V(k,w)$ subject to $k \leq w$ (since $\lim_{k \to 0} h'(k) = \infty$, $k_{opt} > 0$). $V(k,w)$ is twice continuous differentiable in k and strictly concave:

$$\frac{\delta^2 V(k,w)}{\delta k^2} = \left\{ \begin{array}{c} \pi \left(u''(c_E)(\gamma h'(k) - 1)^2 + u'(c_E)\gamma h''(k) \right) \\ +(1-\pi)\left(u''(c_L)(h'(k)-1)^2 + u'(c_L)h''(k) \right) \end{array} \right\} < 0 \tag{A1}$$

We are maximizing a continuous strictly concave function over a compact set $[0,w]$, a maximum k_{opt} exists and is unique. Define $A(k,w) = \frac{\delta V(k,w)}{\delta k}$ and μ, the Lagrange multiplier associated with $k \leq w$.

(i) **First Order Conditions**: $A(k,w) = \frac{\delta V(k,w)}{\delta k} = \mu$; $\left\{ \begin{array}{l} \text{if } k < w \text{ (interior solution) } A(k,w) = 0 \\ \text{if } k = w \text{ (corner solution), } A(k,w) > 0 \end{array} \right.$

(ii) $\frac{\delta A(k,w)}{\delta k} = \frac{\delta^2 V(k,w)}{\delta k^2} < 0$ and $k \leq w \Leftrightarrow A(k,w) \geq A(w,w)$ (with strict inequality if $k < w$).

(iii) $\frac{\delta A(w,w)}{\delta w} < 0$, hence $A(w,w) = 0$ admits a unique solution $w = w^*$ defined by $\frac{u'(\gamma h(w^*))}{u'(h(w^*))} = \frac{(1-\pi)(h'(w^*)-1)}{\pi(1-\gamma h'(w^*))}$.

Hence by (i),(ii) and (iii):

(interior and corner solution): for $w = w^* : A(w^*, w^*) = 0$ and $k_{opt} = w^*$.
(corner solution): for $w < w^* : A(w,w) > 0$ and $k_{opt} = w$.
(interior solution): for $w > w^* : A(k_{opt}, w) = 0 > A(w,w)$ and $k_{opt} < w$ (A2)

$$A(k_{opt}, w) = 0 \Leftrightarrow \frac{u'(c_E)}{u'(c_L)} = \frac{(1-\pi)(h'(k)-1)}{\pi(1-\gamma h'(k))} \tag{A3}$$

and the threshold w^* is defined by $h'(w^*) = \frac{(1-\pi)\gamma^\sigma + \pi}{(1-\pi)\gamma^\sigma + \pi\gamma}$

Properties of the optimal capital choice function $k_{opt}(w)$

For $w \leq w^*, k_{opt}(w) = w$, and the properties for $w \leq w^*$ are obvious.

For $w \geq w^*$, k_{opt} is uniquely defined by $A(k_{opt}, w) = 0$

P1: *$k_{opt}(w)$ is continuous on $[w^*, \infty)$, differentiable in w on $[0, w^*) \cup (w^*, \infty)$ and right and left differentiable at w^*.* $A(k, w)$ is continuously differentiable in k and w.

P2: *$k_{opt}(w)$ is strictly increasing in w*

$$\text{for } w \geq w^* : \frac{\delta k_{opt}(w)}{\delta w} = -\frac{\frac{\delta A(k,w)}{\delta w}}{\frac{\delta A(k,w)}{\delta k}}|_{A(k,w)=0} \tag{A4}$$

$$\frac{\delta A(k,w)}{\delta k} < 0 \text{ by } (A1) \tag{A5}$$
$$\frac{\delta A(k,w)}{\delta w} > 0 \Leftrightarrow \frac{u''(c_L)}{u''(c_E)} < \frac{u'(c_L)}{u'(c_E)} \Leftrightarrow c_L > c_E$$

P3: $k_{opt}(w)$ *is concave* in w and strictly concave for $w \geq w^*$.

P4: $\frac{k_{opt}(w)}{w}$ *is non increasing in w, and strictly decreasing for $w \geq w^*$*

$$\text{for } w \geq w^* : \frac{\delta^2 k_{opt}(w)}{\delta w^2} = \frac{\partial(-\frac{\frac{\delta A(k,w)}{\delta w}}{\frac{\delta A(k,w)}{\delta k}})}{\partial w}|_{A(k,w)=0} \tag{A6}$$

Using (A3) and the properties of the utility function, (A4) can be rewritten as:

$$\frac{\delta k_{opt}(w)}{\delta w} = \frac{-\sigma(c_L - c_E)}{\frac{c_E c_L (1-\gamma) h''(k)}{(1-\gamma h'(k))(h'(k)-1)} - \sigma[c_L(1 - \gamma h'(k)) + c_E(h'(k) - 1)]} = \frac{-\sigma(c_L - c_E)}{D} \tag{A7}$$

hence,

$$\frac{\delta^2 k_{opt}(w)}{\delta w^2} = \frac{\sigma(c_L - c_E)\left[\frac{(c_E+c_L)(1-\gamma)h''(k)}{(1-\gamma h'(k))(h'(k)-1)} - \sigma(1-\gamma)h'(k)\right]}{\mathrm{D}^2} < 0 \tag{A8}$$

At $w = w^*$, $k'^{(-)}_{opt}(w^*) = 1$, while $k'^{(+)}_{opt}(w^*) = \frac{\sigma}{\sigma + \frac{\gamma(1-\beta)h(w^*)h'(w^*)}{w^*(1-\gamma h'(k))(h'(k)-1)}} < 1$

P6 $k_{opt}(w)$ is bounded by $k_{\max} = h'^{-1}\left(\frac{1}{1-\pi+\pi\gamma}\right)$. By construction $c_L > c_E \Leftrightarrow \frac{u'(c_E)}{u'(c_L)} < 1 \Leftrightarrow k_{opt} < h'^{-1}\left(\frac{1}{1-\pi+\pi\gamma}\right) = k_{\max}$ by (A3).hence as $w \to \infty$, $c_E \to c_L$ and $\frac{u'(c_E)}{u'(c_L)} \to 1 \Leftrightarrow h'(k) \to \frac{1}{1-\pi+\pi\gamma}$.

1.2 Growth under Financial Autarky

Since capital fully depreciates after it is used, the connection between the individual problem and the dynamics of the intertemporal model is given by wages of the next generation:

$$w_t = F^a(w_{t-1}) = (1-\beta)(\pi\gamma + 1 - \pi)f(k(w_{t-1})) \tag{A9}$$
$$k_t = k(w_{t-1}) = k_{opt}(w_{t-1})$$

The following proposition characterizes the dynamics of this economy :

Proposition 1.1 *The economy converges towards a unique stable steady state $\bar{w}^a > 0$ and $k(\bar{w}^a)$. The steady state is defined by $F^b(\bar{w}^a) = \bar{w}^a$*

Proof. Define the growth rate of wealth $g\left(w_{t-1}\right)=\frac{F^a\left(w_{t-1}\right)}{w_{t-1}}$.
(i) The growth rate of wealth is decreasing. $1+g(w_{t-1})=\frac{1-\beta}{\beta}(\pi\gamma+1-\pi)f\,'(k(w_{t-1}))\frac{k(w_{t-1})}{w_{t-1}}$. $\frac{k(w_{t-1})}{w_{t-1}}$ is non increasing, $k(w_{t-1})$ is strictly increasing, hence $f\,'(k(w_{t-1})$ is strictly decreasing.
$\lim_{w\to\infty}\frac{k(w_{t-1})}{w_{t-1}}=0$ and $\lim_{w\to\infty}f\,'(k(w_{t-1}))=f'(k_{\max})$, then $\lim_{w\to\infty}1+g(w_{t-1})=0$.
Using l'Hopital Rule, $\lim_{w\to 0}\frac{f(k(w_{t-1}))}{w_{t-1}}=\lim_{w\to 0}f'(k(w_{t-1}))=\infty$.
(ii) The steady state (uniqueness, stability and convergence).
From (i), $\exists!\bar{w}^a$such that $g(\bar{w}^a)=0\Leftrightarrow F^a(\bar{w}^a)=\bar{w}^a$ has a unique non zero root.
Since by definition $g(\bar{w}^a)=0$ and $g(\cdot)$ is strictly decreasing$\Rightarrow\forall w_{t-1}<\bar{w}^a$, $g(w_{t-1})>0$ and $\forall w_{t-1}<\bar{w}^a$, $g(w_{t-1})>0$. Then, the unique steady state is stable.
Since $F^a(w_{t-1})$ is strictly increasing: $w_1>w_0\Longrightarrow F(w_1)>F(w_0)\Leftrightarrow w_2>w_1\Leftrightarrow w_t>w_{t-1}$, $\forall t\geq 0$ (by iteration), thus, convergence is asymptotic. ■

2 Optimal Unconstrained Risk-Sharing Solution

In the unconstrained problem the planner can observe the realization of the idisyncratic liquidty shock of each consumer. At any period t, and for any given level of deposits (wealth $w>0$), the planner chooses $k,\ \lambda,\ i,\ c_E$ and c_L to maximize expected utility of a representative current depositor:

$$\pi u\left(c_E\right)+(1-\pi)u\left(c_L\right)\ \text{ subject to} \tag{1}$$

$$0\leq k\leq w \tag{2}$$

$$0\leq\lambda\leq 1 \tag{3}$$

$$i\geq 0 \tag{4}$$

$$\pi c_E\leq w-k-i+\lambda\gamma h(k) \qquad (\textit{multiplier}\ \ \mu_1) \tag{5}$$

$$\pi c_E+(1-\pi)c_L\leq w-k+\lambda\gamma h(k)+(1-\lambda)h(k) \qquad (\textit{multiplier}\ \ \mu_2) \tag{6}$$

The maximization problem (eqns. 1-6) can be simplified before solving: (a) $k_{opt}>0$ since $h\prime(0)=\infty$; (b) $\lambda<1$, if $\lambda=1$ it would always be possible to increase both c_E and c_L by reducing liquidation of the long technology.

The optimization problem consists to choose $k,\ \lambda,\ i,\ c_L,\ c_E$ to maximize 1 s.t.: (i) $\lambda\geq 0$, (ii) $k\leq w$, 4, 5, 6.

In the solution of the unconstrained risk sharing problem we show: (a) that the first best solution is incentive compatible for all levels of wealth, that is, it satisfies the constraint $c_E\leq c_L$ required in the *covered* and *exposed* banking problems; and (b) the conditions under which the first best rolution is run proof, that is under which conditions $c_E\leq w-k+\gamma h\left(k\right)$.

First Order Conditions. Based on the Kuhn-Tucker conditions:

$$u'\left(c_E\right)=\mu_1+\mu_2 \tag{B1}$$

$$u'(c_L) = \mu_2 \tag{B2}$$

$$(\mu_1 + \mu_2)\,\gamma \leq \mu_2 \text{ with equality if } \lambda > 0 \tag{B3}$$

$$(\mu_1 + \mu_2)\,(1 - \lambda\gamma h'(k)) \leq \mu_2 (1-\lambda) h'(k) \text{ with equality if } w > k \tag{B4}$$

$$\mu_1 \geq 0 \text{ with equlity if } i > 0 \tag{B5}$$

The unconstrained solution is incentive compatible since $u'(c_E) = \mu_1 + u'(c_L)\,, \quad \mu_1 \geq 0 \Rightarrow c_E \leq c_L$

The optimal solution.

Region A $\lambda > 0,\ k = w,\ i = 0:$

$$\lambda > 0 \Rightarrow \frac{u'(c_E)}{u'(c_L)} = \frac{1}{\gamma} \tag{B6}$$

$$\text{(B4) and (B6)} \Rightarrow \gamma h'(w) \geq 1 \Rightarrow \text{region } \mathbf{A} \text{ applies for} : w \in [0, \underline{k}] \tag{B7}$$

$$c_E = \frac{\lambda\gamma h(w)}{\pi} \text{ and } c_L = \frac{(1-\lambda)h(w)}{(1-\pi)} \tag{B8}$$

$$(B8) in (B6) \Rightarrow \lambda_{opt} = \lambda^* \equiv \frac{\pi\gamma^{1/\sigma}}{\pi\gamma^{1/\sigma} + (1-\pi)\gamma}$$

optimal liquidation in this region is constant. Notice that: $\lambda^* \gtreqless \pi \Leftrightarrow \gamma^{\frac{1}{\sigma}} \gtreqless \gamma \ \Leftrightarrow \sigma \gtreqless 1.$

Over region A the UORS solution is run proof if $c_E \leq w - k + \gamma h(k) \Leftrightarrow \lambda \leq \pi \Leftrightarrow \sigma \leq 1$

Region B $\lambda > 0,\ k < w,\ i = 0:$

$$\lambda > 0 \text{ and } k < w \Rightarrow \frac{u'(c_E)}{u'(c_L)} = \frac{1}{\gamma} = \frac{(1-\lambda)\,h'(k)}{1 - \lambda\gamma h'(k)} \tag{B9}$$

this conditions implies a constant optimal choice of capital: $\gamma h'(k) = 1 \ \Leftrightarrow \ k_{opt} = \underline{k}$, and $c_E = \frac{w - \underline{k} + \lambda\gamma h(\underline{k})}{\pi}$ and $c_L = \frac{(1-\lambda)h(\underline{k})}{(1-\pi)}$. The optimal liquidation policy over this region is obtained by substituting the definitions of early and late consumption in the first equality relation of (B9):

$$\lambda_{opt}(w) = \lambda^* - (1-\lambda^*)\,\beta \frac{w - \underline{k}}{\underline{k}}. \tag{B10}$$

Hence optimal liquidation is a decreasing fuction of w. The threshold that defines the upper level of wealth for region B $(\tilde{w})$,is defined by $\lambda_{opt}(\tilde{w}) = 0$ in (B10).

$$\lambda(w) \geq 0 \Leftrightarrow w \leq \tilde{w}^u = \underline{k}\left(1 + \frac{\pi\gamma^{\frac{1}{\sigma}}}{(1-\pi)\,\gamma\beta}\right) \Rightarrow \text{Region } \mathbf{B} \text{ applies for} : w \in [\underline{k}, \tilde{w}^u] \tag{B11}$$

From the definition of c_E and $\lambda(w)$, over Region B, the condition that must satisfy the UORS solution to be run proof is:

$$\begin{aligned} c_E \leq w - \underline{k} + \gamma h(\underline{k}) \quad &\Leftrightarrow \quad (1-\pi)(w - \underline{k}) + (\lambda(w) - \pi)\, h(\underline{k}) \leq 0 \\ &\Leftrightarrow \quad \beta(\lambda^* - \pi)\left(w - \underline{k} + \tfrac{1}{\beta}\underline{k}\right) \leq 0 \\ &\Leftrightarrow \quad \lambda^* \leq \pi \Leftrightarrow \sigma \leq 1 \end{aligned}$$

Region C $\lambda = 0,\ k < w,\ i = 0$:

$$\lambda = 0 \textit{ and } k < w \Rightarrow \frac{u'(c_E)}{u'(c_L)} = h'(k) \tag{B13}$$

$$\text{by (B3)}\ h'(k) \leq \frac{1}{\gamma} \Rightarrow k \geq \underline{k}$$

$$\text{by (B5)}\ h'(k) \leq 1 \Rightarrow k \leq \bar{k}$$

$$h'(k) \to 1 \Rightarrow w \to \hat{w}^u = \bar{k}\left(1 + \frac{\pi}{\beta(1-\pi)}\right) \Rightarrow \text{Region } \mathbf{C} \text{ applies for} : w \in [\tilde{w}^u, \hat{w}^u] \tag{B14}$$

Early and late consumption are given by $c_E = \frac{w - k_{opt}}{\pi},\ c_L = \frac{h(k)}{1-\pi}$. Optimal investment in capital $k_{opt}(w) \in \left[\underline{k}, \bar{k}\right]$ is implicitly defined by (B13) and can be expressed as:

$$\frac{\pi h(k)}{(1-\pi)(w-k)} = h'(k)^{\frac{1}{\sigma}} \tag{B15}$$

Over region C the optimal capital choice is unique $\left(\frac{dA(k,w)}{dk}|_{\tilde{w} \leq w \leq \hat{w}} < 0\right)$:

$$\text{where } A(k,w) = u'\left(\frac{h(k)}{1-\pi}\right)h'(k) - u'\left(\frac{w-k}{\pi}\right) = 0 \Leftrightarrow k = k_{opt}$$

$$\frac{dA(k,w)}{dk} = u'(c_L)\,h''(k) + u''(c_L)\,\frac{h''(k)^2}{1-\pi} + u''(c_E)\,\frac{1}{\pi} < 0$$

From the definition of c_E over Region C, the condition that must satisfy the UORS solution to be run proof is given by:

$$\begin{array}{ccc} c_E \leq w - k + \gamma h(k) & \Leftrightarrow & \frac{\pi h(k)}{(1-\pi)(w-k)} \leq \frac{1}{\gamma} \\ \text{(from B15)}\ \frac{\pi h(k)}{(1-\pi)(w-k)} \leq \frac{1}{\gamma} & \Leftrightarrow & h'(k) \geq \frac{1}{\gamma^\sigma} \end{array}$$

Notice that for $h'(k) \geq \frac{1}{\gamma^\sigma}$ to be consistent with an optimum over this region C ($\gamma h'(k) < 1$), it is necessary that $\sigma \leq 1$.

Define k_{rp} such that $h'(k_{rp}) = \frac{1}{\gamma^\sigma}$ and from (B15) we have that the threshold for the satisfaction of the run preventive constraint is given by: $w_{rp} = k_{rp}\left(1 + \frac{\pi\gamma}{\beta(1-\pi)\gamma^\sigma}\right)$, thus:

$$\begin{array}{ccc} c_E \leq w - k + \gamma h(k) & \Leftrightarrow & \sigma \leq 1 \text{ and } w \leq w_{rp} = k_{rp}\left(1 + \frac{\pi\gamma}{\beta(1-\pi)\gamma^\sigma}\right) \\ \text{where } k_{rp} \text{ is defined by} & \Leftrightarrow & h'(k_{rp}) = \frac{1}{\gamma^\sigma} \end{array} \tag{B16}$$

Region D: $\lambda = 0,\ k < w\ , i > 0 \Rightarrow \mu_1 = 0$

$$\lambda = 0 \text{ and } k < w \ \Rightarrow \frac{u'(c_E)}{u'(c_L)} = h'(k) \tag{B17}$$

$$\mu_1 = 0 \Rightarrow c_E = c_L = c \Leftrightarrow h'(k) = 1 \Leftrightarrow k_{opt} = \bar{k} \tag{B18}$$

and optimal consumption is: $c = w - \bar{k} + h(\bar{k})$. Thus since $c_E = \frac{w - \bar{k} - i}{\pi} = w - \bar{k} + h(\bar{k}) \Rightarrow$

$$i^u = (1-\pi)\left(w - \bar{k}\right) - \pi h\left(\bar{k}\right) \tag{B19}$$

Since $c = c_E = w - \bar{k} + h(\bar{k}) > w - \bar{k} + \gamma h(\bar{k}) = c_R$ the UORS over region D cannot satisfy the run preventive constraint.

■

3 The Optimal Covered Banking Solution

In this section, we present the optimal solution of covered banking. The optimal solution defines four region (A to D), after presenting the solution of each region, we present the marginal cost of run prevention.

A covered bank solves a similar problem that the unconstraned optimal risk sharing planner but with two additional restrictions. It must satisfy the incentive compatibility constraint $c_E \leq c_L$, and the run preventive constraint $c_E \leq c_R = w - k + \gamma h(k)$. A covered bank chooses k, λ, i, c_E and c_L to maximize expected utility of a representative current depositor:

$$\max_{\lambda,k,c_E,c_L} \quad \pi u(c_E) + (1-\pi)u(c_L) \text{ subject to} \tag{7}$$

$$0 \leq k \leq w \tag{8}$$

$$0 \leq \lambda \leq 1 \tag{9}$$

$$i \geq 0 \tag{10}$$

$$c_E \leq c_L \tag{11}$$

$$\pi c_E \leq w - k - i + \lambda\gamma h(k) \qquad (\textit{multiplier } \mu_1) \tag{12}$$

$$\pi c_E + (1-\pi)c_L \leq (1-\lambda)h(k) + w - k + \lambda\gamma h(k) \qquad (\textit{multiplier } \mu_2) \tag{13}$$

$$c_E \leq w - k + \gamma h(k) \qquad (\textit{multiplier } \mu_3) \tag{14}$$

As in the unconstrained problem we solve without impossing the incentive compatibility constraint (11), and show that the solution satisfies it. The first order conditions of the problem are:

$$u'(c_E) = \mu_1 + \mu_2 + \frac{\mu_3}{\pi} \tag{C1}$$

$$u'(c_L) = \mu_2 \tag{C2}$$

$$(\mu_1 + \mu_2)\gamma \leq \mu_2 \quad \text{with equality if } \lambda > 0 \tag{C3}$$

$$(\mu_1 + \mu_2)(1 - \lambda\gamma h'(k)) \leq \mu_2(1-\lambda)h'(k) + \mu_3(\gamma h'(k) - 1) \quad \text{with equality if } w > k \tag{C4}$$

$$\mu_1 \geq 0 \;\; \textit{with equality if } i > 0 \tag{C5}$$

Notice that since $\mu_1, \; \mu_3 \geq 0 \Rightarrow$ the incentive compatibility constraint $c_E \leq c_L$ is always satisfied. In addition, notice that if the run preventive constraint is not binding $c_E < w - k + h(k) \Rightarrow \mu_3 = 0$ the optimality conditions are identical to the unconstrained optimal conditions, and the solution will be identical. We have shown that for $\sigma \leq 1$ and $w \leq w_{rp}$ the unconstrained problem satisfies the run preventive constraint. Therefore, for $\sigma > 1$, for all levels of wealth, and for $\sigma \leq 1$ for $w > w_{rp}$ the run preventive constraint binds. For these cases we have then the following additional condition:

$$c_E = \frac{w - k - i + \lambda\gamma h(k)}{\pi} = w - k + \gamma h(k) \tag{C6}$$

The marginal cost on liquidity insurance of the run preventive constraint is measured as a marginal rate of substitution $\left(\frac{\mu_3}{\mu_2}\right)$, defined by the marginal cost of in terms of early consumption lost by keeping the constraint (μ_3) relative to the marginal gain in terms of late consumption (μ_2).

Region A $k = w,\ \lambda > 0,\ i = 0,\ c_E = c_R$

$$c_E = c_R \text{ and } k = w \Rightarrow \frac{\lambda\gamma h(w)}{\pi} = \gamma h(w) \Leftrightarrow \lambda = \pi \tag{C7}$$

$$\begin{aligned} c_L &= \frac{(1-\lambda)h(w)}{1-\pi} = h(w) \Rightarrow c_E = \gamma c_L \\ &\Rightarrow \frac{u'(c_E)}{u'(c_L)} = \frac{1}{\gamma^\sigma} \end{aligned} \tag{C8}$$

From (C1), (C4) and (15) we have that this conditions hold for:

$$(\gamma h'(k) - 1)\left(1 + \gamma\pi\left(\tfrac{1}{\gamma^\sigma} - \tfrac{1}{\gamma}\right)\right) \geq 0$$

$$\begin{aligned} \text{since } \sigma &\geq 1 \Rightarrow \frac{1}{\gamma^\sigma} > \frac{1}{\gamma} \Rightarrow h'(k) \geq \frac{1}{\gamma} \Rightarrow \\ \text{region } \mathbf{A} \text{ applies for} &: w \in [0, \underline{k}] \end{aligned}$$

From (C1), (C2) and (C3) in region A, the marginal cost on liquidity insurance of the run preventive constraint $\frac{\mu_3}{\mu_2}$ is:

$$\frac{\mu_3}{\mu_2} = \pi\left(\frac{1}{\gamma^\sigma} - \frac{1}{\gamma}\right) \quad \text{constant} \tag{C9}$$

Region B $k < w,\ \lambda > 0,\ i = 0,\ c_E = c_R$

$$\begin{aligned} \text{From } i &= 0 \text{ and (C5): } c_E = c_R \Leftrightarrow \\ \lambda(w) &= \pi - (1-\pi)\beta\frac{w-k}{h(k)} \end{aligned} \tag{C10}$$

$$\begin{aligned} \text{From (15)} &: c_E = \frac{1-\lambda}{1-\pi}\gamma h(k) \\ \text{and since and } c_L &= \frac{(1-\lambda)h(k)}{1-\pi} = h(w) \Rightarrow c_E = \gamma c_L \Rightarrow \\ \frac{u'(c_E)}{u'(c_L)} &= \frac{1}{\gamma^\sigma} \end{aligned} \tag{C11}$$

$$\text{from (C1 to C4) and (15): } h'(k) = \frac{1}{\gamma} \Rightarrow k = \underline{k}$$

$$\begin{aligned} k &= \underline{k}, \text{ and (15): } \lambda(w) \geq 0 \Leftrightarrow w \leq \tilde{w}^c = \underline{k}\left(1 + \frac{\pi}{\beta(1-\pi)}\right) \\ \text{region } \mathbf{B} \text{ applies for} &: w \in [\underline{k}, \tilde{w}^c] \end{aligned}$$

The marginal cost on liquidity insurance of the run preventive constraint $\frac{\mu_3}{\mu_2}$ is:

$$\frac{\mu_3}{\mu_2} = \pi\left(\frac{1}{\gamma^\sigma} - \frac{1}{\gamma}\right) \quad \text{constant} \tag{C12}$$

Region C $k < w;\ \lambda = 0,\ i = 0,\ c_E = c_R$

$$\begin{aligned}
\text{From } i &= 0,\ \lambda = 0 \text{ and (C5): } c_E = c_R \Rightarrow \\
c_E &= \gamma \frac{h(k)}{1-\pi} \text{ and } c_L = \frac{h(w)}{1-\pi} \Rightarrow c_E = \gamma c_L \Rightarrow \\
\frac{u'(c_E)}{u'(c_L)} &= \frac{1}{\gamma^\sigma} \qquad \text{(C13)}
\end{aligned}$$

The optimal choice of capital is implicitly defined by this marginal rate of substitution:

$$(1-\pi)(w-k) - \gamma\pi h(k) = 0 \qquad \text{(C14)}$$

Over region C the optimal capital choice is unique $\left(\frac{dA(k,w)}{dk}|_{\tilde{w}\le w\le \hat{w}} < 0\right)$:

$$\begin{aligned}
\text{where } A(k,w) = u'\left(\frac{h(k)}{1-\pi}\right)h'(k) - u'\left(\frac{w-k}{\pi}\right) = 0 \Leftrightarrow k = k_{opt} \\
\frac{dA(k,w)}{dk} = u'(c_L)\,h''(k) + u''(c_L)\frac{h''(k)^2}{1-\pi} + u''(c_E)\frac{1}{\pi} < 0
\end{aligned}$$

From (C1), (C2) and the MRS $\frac{u'(c_E)}{u'(c_L)} = \frac{1}{\gamma^\sigma}$

$$\frac{\mu_3}{\mu_2} = \pi\left(\frac{1}{\gamma^\sigma} - \frac{\mu_1+\mu_2}{\mu_2}\right) \qquad \text{(C15)}$$

From (C4)

$$\frac{\mu_1+\mu_2}{\mu_2} = h'(k) - \frac{\mu_3}{\mu_2}\left(1-\gamma h'(k)\right) \qquad \text{(C16)}$$

From (C5) $i\mu_1 = 0$, then the limit of this region is defined when $\mu_1 \to 0$. From (C15), (C16) and $\mu_1 = 0$ the capital level at the boundary of the region is:

$$\begin{aligned}
1 &= h'\left(\hat{k}^c\right) - \pi\left(\frac{1}{\gamma^\sigma} - 1\right)\left(1-\gamma h'\left(\hat{k}^c\right)\right) \Rightarrow \\
h'\left(\hat{k}^c\right) &= \frac{\pi + (1-\pi)\gamma^\sigma}{\pi\gamma + (1-\pi\gamma)\gamma^\sigma} \qquad \text{(C17)}
\end{aligned}$$

and from (C14):

$$\begin{aligned}
\hat{w}^c &= \hat{k}^c\left(1 + \frac{\gamma\pi}{\beta(1-\pi)}h'\left(\hat{k}^c\right)\right) \qquad \text{(C18)} \\
\text{region } \mathbf{C} \text{ applies for} &: \quad w \in [\tilde{w}^c, \hat{w}^c]
\end{aligned}$$

- Substituting (C16) into (C15) we obtain the marginal cost on liquidity insurance of run prevention

$$\frac{\mu_3}{\mu_2} = \frac{\pi\left(\frac{1}{\gamma^\sigma} - h(k)\right)}{1-\pi\left(1-\gamma h'(k)\right)} \qquad \text{(C19)}$$

Since $\sigma > 1 \Rightarrow \frac{1}{\gamma^\sigma} > \frac{1}{\gamma}$ and since $w > \hat{w}^c$, $h'(k) < \frac{1}{\gamma}$. Therefore, the marginal cost of run prevention is positive. This marginal cost is increasing with the level of wealth over this region since:[1]

$$\frac{d\frac{\mu_3}{\mu_2}}{dw} = \frac{-h''(k)}{1-\pi\left(1-\gamma h'(k)\right)}\left(1+\pi\gamma\frac{\mu_3}{\mu_2}\right)\frac{dk}{dw} > 0 \qquad \text{(C20)}$$

[1] $\frac{dk}{dw} = \frac{1-\pi}{1-\pi+\pi\gamma h'(k)} > 0$

Region D $k < w;\ \lambda = 0,\ i > 0,\ c_E = c_R$

$$\begin{aligned} \lambda &= 0,\ c_E = c_R \Rightarrow \frac{w-k-i}{\pi} = w - k + \gamma h(k) \\ i &= (1-\pi)(w-k) - \pi\gamma h(k) \end{aligned} \tag{C21}$$

(C15), (C16) hold with $\mu_1 = 0 \Rightarrow \frac{u'(c_E)}{u'(c_L)} = \frac{h'(k)-1+\pi\left(1-\gamma h'(k)\right)}{\pi(1-\gamma h'(k))}$

$$\frac{u'(c_E)}{u'(c_L)} = \frac{1-\pi}{\pi} \frac{\left(\frac{1-\pi\gamma}{1-\pi} h'(k) - 1\right)}{1-\gamma h'(k)} \tag{C22}$$

The optimal choice of capital is implicitly defined by the definitions of i (15) and of the MRS (C22)

$$\left(\frac{\frac{1-\pi\gamma}{1-\pi} h(k) + w - k}{w - k + \gamma h(k)}\right)^{\sigma} = \frac{1-\pi}{\pi} \frac{\left(\frac{1-\pi\gamma}{1-\pi} h'(k) - 1\right)}{1-\gamma h'(k)} \tag{C23}$$

which implies that:

$$\lim_{w->\infty} h'(k) = 1 \Leftrightarrow k^c(w) \to k^u(w) \tag{C23b}$$

- The marginal cost on liquidity insurance of run prevention is obtained by substituting (C16) into (C15):

$$\frac{\mu_3}{\mu_2} = \frac{h(k) - 1}{1 - \gamma h'(k)} \tag{C24}$$

that is decreasing with the level of wealth:

$$\frac{d\frac{\mu_3}{\mu_2}}{dw} = \frac{h''(k)}{1-\gamma h'(k)} \left(1 + \gamma \frac{\mu_3}{\mu_2}\right) \frac{dk}{dw} < 0 \tag{C25}$$

Furthermore by combining (C23b) and (C24):

$$\lim_{w\to\infty} \frac{\mu_3}{\mu_2} = 0 \tag{C26}$$

4 The Optimal Exposed Banking Solution

In this section, we present the optimal solution of optimal exposed banking. An exposed bank chooses $k,\ \lambda,\ i,\ c_E$ and c_L to maximize expected utility of a representative current depositor:

$$\max_{\lambda,k,c_E,c_L} \quad (1-q)\,\pi u(c_E) + (1-\pi) u(c_L) + q u(c_R) \ \text{ subject to} \tag{15}$$

$$0 \le k \le w \tag{16}$$

$$0 \le \lambda \le 1 \tag{17}$$

$$i \ge 0 \tag{18}$$

$$c_E \le c_L \tag{19}$$

$$\pi c_E \le w - k - i + \lambda\gamma h(k) \qquad (\textit{multiplier} \ \ \phi_1) \tag{20}$$

$$\pi c_E + (1-\pi) c_L \le (1-\lambda) h(k) \ + w - k + \lambda\gamma h(k) \qquad (\textit{multiplier} \ \ \phi_2) \tag{21}$$

$$c_R = w - k + \gamma h(k) \qquad (\textit{multiplier} \ \ \phi_3) \tag{22}$$

In a similar way to the unconstrained problem and the covered bankikng problem we solve without the incentive compatibility constraint (19) and show that the solution satisfies it.

First Order Conditions

$$(1-q)\,u'(c_E) = \phi_1 + \phi_2 \tag{D1}$$

$$(1-q)\,u'(c_L) = \phi_2 \tag{D2}$$

$$qu'(c_R) = \phi_3 \tag{D3}$$

$$(\phi_1+\phi_2)\,\gamma \le \phi_2 \qquad \text{with equality if } \lambda > 0 \tag{D4}$$

$$(\phi_1+\phi_2)\,(1-\lambda\gamma h'(k)) \le \phi_2(1-\lambda)h'(k) + \phi_3\,(\gamma h'(k)-1) \quad \text{with equality if } w > k \tag{D5}$$

$$\phi_1 \ge 0 \text{ with equality if } i > 0 \tag{D5b}$$

Since $\phi_1 \ge 0$ the incentive compatibility constraint is satisfied.

The optimal solution.

Region A $\lambda > 0,\ k = w,\ i = 0:$

$$\lambda > 0 \Rightarrow \frac{u'(c_E)}{u'(c_L)} = \frac{1}{\gamma} \tag{D6}$$

$$c_E = \frac{\lambda\gamma h(w)}{\pi},\ c_L = \frac{(1-\lambda)h(w)}{(1-\pi)},\ \text{and } c_R = \gamma h\,(w) \tag{D7}$$

$$(D6)\ in\ (D7) \Rightarrow \lambda_{opt} = \lambda^* \equiv \frac{\pi\gamma^{1/\sigma}}{\pi\gamma^{1/\sigma} + (1-\pi)\gamma} \tag{D8}$$

From (D5 and D6),

$$\begin{aligned}
\frac{1}{\gamma} &\le h'(k)\left(1+\gamma\frac{(1-q)}{q}\frac{u'(c_R)}{u'(c_L)}\right) - \frac{(1-q)}{q}\frac{u'(c_R)}{u'(c_L)} \\
h'(k) &\ge \frac{\frac{1}{\gamma} + \frac{(1-q)}{q}\frac{u'(c_R)}{u'(c_L)}}{1+\gamma\frac{(1-q)}{q}\frac{u'(c_R)}{u'(c_L)}} \\
&\Rightarrow \gamma h'(w) \ge 1 \Rightarrow \text{region } \mathbf{A} \text{ applies for} : w \in [0, \underline{k}]
\end{aligned} \tag{D9}$$

Hence the conditions for region A are the same as in the unconstrained optimal risk sharing.

Region B $k < w,\ \lambda > 0,\ i = 0$

$$\lambda > 0 \Rightarrow \frac{u'(c_E)}{u'(c_L)} = \frac{1}{\gamma} \tag{D10}$$

$$c_E = \frac{w-k+\lambda\gamma h(w)}{\pi},\ c_L = \frac{(1-\lambda)h(w)}{(1-\pi)},\ \text{and } c_R = w-k+\gamma h\,(w) \tag{D11}$$

From $(D10)$ and (D11)

$$\lambda_{opt}(w) = \lambda^* - (1-\lambda^*)\,\beta\frac{w-k}{k} \tag{D12}$$

The upper threshold of region B is defined by $\lambda \to 0$, using (D5), (D6) and (D11)

$$(1-\lambda)\gamma h'(k) - \frac{q}{1-q}\frac{u'(c_r)}{u'(c_L)}\gamma(1-\gamma h'(k)) = (1-\lambda\gamma h'(k))$$

$$\begin{aligned} \lambda &\to 0 \Rightarrow -\frac{q}{1-q}\frac{u'(c_r)}{u'(c_L)}\gamma(1-\gamma h'(k)) = 1-\gamma h'(k) \Leftrightarrow \\ h'(k) &= \frac{1}{\gamma} \Rightarrow k = \underline{k} \end{aligned}$$

by $\lambda_{opt}(\tilde{w}) = 0$ in (D12).

$$\lambda(w) \geq 0 \Leftrightarrow w \leq \tilde{w}^e = \tilde{w}^u = \underline{k}\left(1 + \frac{\pi\gamma^{\frac{1}{\sigma}}}{(1-\pi)\gamma\beta}\right) \Rightarrow \text{Region } \mathbf{B} \text{ applies for} : w \in [\underline{k}, \tilde{w}^u] \tag{D13}$$

Region C $k < w,\ \lambda = 0,\ i = 0$

$$\frac{c_L}{c_E} = \frac{\pi h(k)}{(1-\pi)(w-k)} \quad and \quad \frac{c_L}{c_R} = \frac{h(k)}{(1-\pi)(w-k+\gamma h(k))} \tag{D14}$$

From (D1 to D3) and (D5),

$$\frac{u'(c_E)}{u'(c_L)} = h'(k) - \frac{q}{1-q}\frac{u'(c_R)}{u'(c_L)}(1-\gamma h'(k)) \tag{D15}$$

Optimal investment in capital is implicitly defined by (D15) with the values of the ratios of consumption defined in (D14).

Let the upper threshold of region C be $\hat{w}^e$, it is defined by (D5b) as $\phi_1 \to 0$. Notice that in this case $c_E \to c_L \Leftrightarrow \pi h(k) = (1-\pi)(w-k)$. at $w = \hat{w}^e$ then at the threshold (D14) becomes:

$$\frac{c_L}{c_R} = \frac{1}{\pi + \gamma(1-\pi)}$$

$$\begin{aligned} and\ (D15) \quad &: \quad h'\left(\hat{k}^e\right) = \frac{1 + \frac{q}{1-q}\left(\frac{1}{\pi+\gamma(1-\pi)}\right)^{\sigma}}{1 + \frac{q}{1-q}\gamma\left(\frac{1}{\pi+\gamma(1-\pi)}\right)^{\sigma}} \\ h'\left(\hat{k}^e\right) &= \frac{q + (1-q)(\pi+(1-\pi)\gamma)^{\sigma}}{q\gamma + (1-q)(\pi+(1-\pi)\gamma)^{\sigma}} > 1 \end{aligned} \tag{D16}$$

and the threshold is defined by:

$$\hat{w}^e = \hat{k}^e\left(1 + \frac{\pi h'\left(\hat{k}^e\right)}{\beta(1-\pi)}\right) \text{Region } \mathbf{C} \text{ applies for} : w \in [\tilde{w}^u, \hat{w}^e] \tag{D17}$$

Region D $k < w,\ \lambda = 0,\ i > 0$

When the bank starts using excess liquidty $i > 0 \Rightarrow \phi_1 = 0$, the bank provides full liquidity insurance

$$\begin{aligned} \frac{u'(c_E)}{u'(c_L)} &= 1 \Rightarrow \\ c_E &= c_L = w - k + h(k) \end{aligned}$$

therefore since $c_E = \frac{w-k-i}{\pi}$ and $c_L = \frac{h(k)+i}{1-\pi} \Rightarrow$ excess liquidty is defined as

$$i = (1-\pi)(w-k) - \pi h(k) \tag{D18}$$

An examination of the first order conditions show that (D15) still holds. Hence optimal investment in capital is implicitly defined by:

$$(1-q)(h'(k)-1)(w-k+\gamma h(k))^{\sigma} = q(w-k+h(k))^{\sigma}(1-\gamma h'(k)) \tag{D19}$$